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Duality for projective curves

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Abstract. This paper studies the strict dual of a projective algebraic curve, mainly in positive characteristic. Inclusion relations among the osculating spaces of the dual and the duals of those of the curve are obtained and shown to be optimal in several cases. As a consequence, a characterization of the non-reflexive curves that coincide with their bidual is obtained.

1. Introduction

The duality theory of plane curves has been studied for a long time. Duality for conics is behind many classical results, for example Brianchon's theorem. The theory began to be developed for arbitrary plane curves by Plücker as soon as he introduced the concept of homogeneous coordinates. He, in particular, used the dual of a plane curve to prove the now famous Plücker formulae. A fundamental property of plane curves in characteristic zero is that they equal their dual's dual, as was first proved by Monge (see, e.g. [11]). Recently, the theory of duality of plane curves has been reconsidered, specially in positive characteristic (see [7] and [10]).

The extrinsic geometry of algebraic curves in higher dimensional projective spaces was also considered classically. The theory, in characteristic zero, was essentially completed by the Italian school (see [2] and [16]). A natural object in this study is the (strict) *dual*) of an algebraic curve $X \subseteq \mathbf{P}^n$, which is defined as the closure of the set of osculating hyperplanes to the curve X, and was used by Cayley and Veronese to generalize the Plücker formulae (see [13] for a modern treatment and [12] for another approach). A basic result is that the intermediate osculating spaces of the dual curve coincide with the duals of those of the curve (with the appropriate dimensions) and, in particular, a curve is equal to its dual's

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dual. All this can fail in positive characteristic p, but the result still holds when $p > n$ and X is *classical*, that is, no hyperplane touches X at a generic point with intersection multiplicity greater than n. The proof of this can be given by noticing that the proof in characteristic zero (as given e.g. in [2] pp. 449-451) also works under the above hypotheses (see also the proof of Theorem 5.1 in [13] and Remark 2 here).

The purpose of this paper is to investigate the dual of an algebraic curve and the relations among the intermediate osculating spaces of the curve and those of its dual. We characterize when the curve equals its *bidual* (i.e. its dual's dual). The characterization being that the osculating hyperplane to X at a generic point P should contain a certain osculating space to X at $F(P)$, where F is the map obtained by composing the Gauss map of the curve with that of its dual. The result is particularly nice for smooth plane curves. In this case we show that the non-reflexive curves that equal their biduals are precisely the *Frobenius-non-classical* curves, that is, the curves for which the image under the Frobenius map of a point lies on the tangent line to the curve at that point. These curves first turned up in [17] where it is given a geometric method to get upper bounds for the number of rational points of curves over finite fields.

There are several examples throughout the paper that illustrate the results and the necessity of the hypotheses. We state the properties of the examples but, as a rule, we omit the verifications. The computations involved are all straightforward, albeit rather long in some cases. Nevertheless, they were all done by hand.

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2. Preliminaries

We will always work over an algebraically closed field of characteristic p, $p = 0$ or a prime number.

Let $X \subseteq \mathbf{P}^n$ be an irreducible algebraic curve, which we assume to be *non-degenerate*, that is, not contained in a hyperplane. The *order-sequence* of $X \subseteq \mathbf{P}^n$ is defined to be the possible intersection multiplicities of X with hyperplanes at a generic point of X. The order-sequence consists of $n+1$ non-negative integers which we denote by $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_n$. It is always the case that $\varepsilon_0 = 0$, $\varepsilon_1 = 1$ (see [15] and [17]). When $\varepsilon_i = i$, $i = 0, \ldots, n$, X is said to be *classical*. All curves are classical in characteristic zero.

If X is as above, denote by $L_P^{(r)}X$, $r = 0, \ldots, n-1$, the linear space which is the intersection of all hyperplanes H of $\mathbf{P}^n$ for which the intersection multiplicity of X and H at the generic point P is greater than ε_r. The space $L_P^{(r)}X$ is called the *r-th osculating space* to X at P and has dimension r. In the special case $r = 1$, $L_P^{(1)}X = T_PX$ is the *tangent line* to X at P and, in the case $r = n-1$, $L_P^{(n-1)}X$ is the *osculating hyperplane* to X at P.

Denote by $D^{(j)}$, $j = 0, 1, 2, \ldots$, a family of Hasse derivatives with respect to some separating variable on X. We shall need the following properties of the Hasse derivatives (see [6] and [7]). For any functions x, y on X, we have:

- $D^{(j)}xy = \sum_{r=0}^{j} D^{(r)}x \cdot D^{(j-r)}y$.
- $D^{(r)}D^{(j)}x = \binom{r+j}{r} D^{(r+j)}x$.
- If q is a power of $p > 0$, then $D^{(j)}x^q = \begin{cases} (D^{(j/q)}x)^q, & \text{if } j \equiv 0(q). \\ 0, & \text{otherwise}. \end{cases}$

If $x_0, \ldots, x_n$ are functions on X such that the embedding $X \subseteq \mathbf{P}^n$ is given by $(x_0 : \ldots : x_n)$, then $L_P^{(r)}X$ is spanned by $(D^{(j)}x_0 : \cdots : D^{(j)}x_n)$, $0 \le j \le \varepsilon_r$. From this it follows that the osculating hyperplane to X at P is given by $\sum \alpha_i X_i = 0$, where $\alpha_0, \ldots, \alpha_n$ are functions on X that satisfy

$$\sum_{i=0}^{n} \alpha_i \cdot D^{(j)}x_i = 0, \qquad 0 \le j \le \varepsilon_{n-1}. \tag{0}$$

Let $(\mathbf{P}^n)'$ be the dual projective space (i.e. the space of hyperplanes of $\mathbf{P}^n$). The *Gauss map* $\gamma: X \to (\mathbf{P}^n)'$ is the rational map defined at a generic point P by $\gamma(P) = \left(L_P^{(n-1)}X\right)'$, where for a linear space V we denote by V' its dual. In the above choice of coordinates the Gauss map γ is then given by $(\alpha_0 : \ldots : \alpha_n)$. The closure of the image $\gamma(X)$ is denoted here by X' and is called the *dual* of X. We abuse notation and denote also by γ the map $\gamma: X \to X'$.

Let $deg_i\varphi$ denote the degree of inseparability of a non-constant rational map φ between algebraic curves. We will use constantly throughout this paper the following result due to Hefez and Kakuta.

Theorem. (([8]).) Let $\gamma: X \to X'$ be the Gauss map of the curve $X \subseteq \mathbf{P}^n$. Then, $deg_i\gamma$ is the highest power of the characteristic $p > 0$ dividing ε_n.

We also use frequently the fact that if, for an integer ε, $\binom{\varepsilon_r}{\varepsilon} \not\equiv 0(p)$ for some order ε_r of X, then ε is also an order of X (see [14] and [17]). Here $\binom{\varepsilon_r}{\varepsilon}$ denotes the binomial coefficient. We denote by $[x]$ the integral part of the real number x. Finally, in case of characteristic $p = 0$, $q = 1$ is to be considered as a power of p.

3. The main results

Our basic result is the following theorem which relates the osculating spaces of a projective curve with those of its dual. The result is stated more generally for projective morphisms, not necessarily birational, in view of the applications in §4 to questions of biduality.

Theorem 1. *Let X be a reduced, irreducible, complete algebraic curve and let $\varphi_1: X \to \mathbf{P}^n$, $\varphi_2: X \to (\mathbf{P}^n)'$ be rational maps with non-degenerate images, satisfying $deg_i\varphi_2 = q deg_i\varphi_1$, for some $q \geq 1$. Let $\delta_0 < \cdots < \delta_n$ and $\mu_0 < \cdots < \mu_n$ be the order-sequences of $\varphi_1(X)$ and $\varphi_2(X)$, respectively, and assume that $\delta_n = mq$ for some integer $m \geq 1$. Define the integer d, $0 \leq d \leq n$, by $\mu_d < m \leq \mu_{d+1}$. Finally, let γ_1 be the Gauss map of $\varphi_1(X)$. For $\varphi_2 = \gamma_1 \circ \varphi_1$ to be valid it is necessary and sufficient that*

$$L^{(d)}_{\varphi_2(P)}\varphi_2(X) \subseteq \varphi_1(P)'$$

holds for P generic. Moreover, if this is the case, then $m = \mu_{d+1} \not\equiv 0(p)$ and $L^{(d+1)}_{\varphi_2(P)}\varphi_2(X) \not\subseteq \varphi_1(P)'$ for P generic. Also, defining s_j (for $j = 0, \ldots, n-1$) by $\mu_{s_j} < m - [\delta_j/q] \leq \mu_{s_j+1}$, we have that

$$L^{(s_j)}_{\varphi_2(P)}\varphi_2(X) \subseteq \left(L^{(j)}_{\varphi_1(P)}\varphi_1(X)\right)'$$

holds for P generic.

Proof. Choose coordinates in $(\mathbf{P}^n)'$ and $\mathbf{P}^n$. Then there are functions $x_0, \ldots, x_n, y_0, \ldots, y_n$ on X such that $\varphi_1 = (x_0^{q_1} : \cdots : x_n^{q_1})$ and $\varphi_2 = (y_0^{q_2} : \cdots : y_n^{q_2})$, where $q_j = \deg_i\varphi_j$, $j = 1, 2$. Now, $\varphi_2 = \gamma_1 \circ \varphi_1$ if and only if

$$\sum_{i=0}^{n} y_i^q \cdot D^{(j)}x_i = 0, \quad \text{for} \quad 0 \leq j \leq \delta_{n-1}. \tag{1}$$

Also, the condition $L^{(s_j)}_{\varphi_2(P)}\varphi_2(X) \subseteq \left(L^{(j)}_{\varphi_1(P)}\varphi_1(X)\right)'$ for P generic, is expressed by

$$\sum_{i=0}^{n}(D^{(r)}y_i)^q \cdot D^{(s)}x_i = 0, \quad \text{for} \quad 0 \le r \le \mu_{s_j} \quad \text{and} \quad 0 \le s \le \delta_j. \quad (2_j)$$

By definition $s_0 = d$, hence what we want to show is that (1) is equivalent to (2_0). Note that (1) follows from the equations in (2_{n-1}) by taking $r = 0$. Let us now show that (1) implies (2_j), for $0 \le j \le n-1$, and that (2_0) also implies (2_j), for $0 \le j \le n-1$.

Assume that (1) holds. We begin by showing, by induction on h, that $\sum_{i=0}^{n}(D^{(h)}y_i)^q \cdot D^{(jq)}x_i = 0$ if $h + j < m$. The case $h = 0$ follows from (1) since $jq < mq = \delta_n$. Now, if $j < m - h \le m$,

$$0 = D^{(hq)}\left(\sum_{i=0}^{n} y_i^q \cdot D^{(jq)}x_i\right) = \sum_{t=0}^{h}\binom{j+t}{t}\sum_{i=0}^{n}(D^{(h-t)}y_i)^q \cdot D^{((j+t)q)}x_i.$$

All summands of the last expression, for $t \ge 1$, vanish by the induction hypothesis and therefore the summand with $t = 0$ also vanishes, as desired.

Now, if $\ell < q$ and $h + j < m$, then we have

$$0 = D^{(\ell)}\left(\sum_{i=0}^{n}(D^{(h)}y_i)^q \cdot D^{(jq)}x_i\right) = \sum_{i=0}^{n}(D^{(h)}y_i)^q \cdot D^{(jq+\ell)}x_i,$$

which implies (2_j), for $0 \le j \le n-1$. One can similarly prove that (2_0) implies (2_j), for $0 \le j \le n-1$, by induction on j on applying the operator $D^{(jq)}$ to (2_0).

It remains to be shown, on the presence of (1), that $m = \mu_{d+1} \not\equiv 0(p)$ and that $L^{(d+1)}_{\varphi_2(P)}\varphi_2(X) \not\subseteq \varphi_1(P)'$ for P generic.

First, $\deg_i\gamma_1$ is the highest power of p dividing δ_n and also $\deg_i\gamma_1 = \deg_i\varphi_2/\deg_i\varphi_1 = q$, hence $m \not\equiv 0(p)$.

If m is not an order of $\varphi_2(X)$ or if $m = \mu_{d+1}$ and $L^{(d+1)}_{\varphi_2(P)}\varphi_2(X) \subseteq \varphi_1(P)'$ for P generic, then (2_0) holds for $r \le m$. This implies, by an argument similar to the above, that (1) holds for $j \le mq = \delta_n$, which is an absurd. This completes the proof of the theorem. □

Remark 1. From the set of equations (2_j) it also follows that $L^{(r_j)}_{\varphi_1(P)}\varphi_1(X) \subseteq$

$\left(L^{(j)}_{\varphi_2(P)}\varphi_2(X)\right)'$, where r_j is defined by $\delta_{r_j} < (m-\mu_j)q \le \delta_{r_j+1}$, as the reader can easily check.

A natural question that arises from the above theorem is whether s_j is the largest integer s with the property that $L^{(s)}_{\varphi_2(P)}\varphi_2(X) \subseteq (L^{(j)}_{\varphi_1(P)}\varphi_1(X))'$ for P generic. Although this is not true in all cases, it is true under certain hypotheses, as shown by the next result.

Theorem 2. *Notation and hypotheses as in Theorem 1. Assume further that* $\varphi_2 = \gamma_1 \circ \varphi_1$. *For each* $j = 0, 1, \ldots, n-1$ *and for* P *generic, both* $\mu_{s_j+1} = m - [\delta_j/q]$ *and* $L^{(s_j+1)}_{\varphi_2(P)}\varphi_2(X) \not\subset (L^{(j)}_{\varphi_1(P)}\varphi_1(X))'$ *hold if and only if we have* $\binom{m}{[\delta_j/q]} \not\equiv 0(p)$.

Proof. Let, for each $r = 0, \ldots, d+1$,

$$b_r = \sum_{i=0}^{n}(D^{(\mu_r)}y_i)^q \cdot D^{(q(m-\mu_r))}x_i,$$

where x_i (resp. y_i) are coordinates for φ_1 (resp. φ_2). We claim that

$$b_r \neq 0 \quad \text{if and only if } \binom{m}{\mu_r} \not\equiv 0(p).$$

Let us see how this implies the theorem. Suppose first that $\mu_{s_j+1} = m - [\delta_j/q]$ and $L^{(s_j+1)}_{\varphi_2(P)}\varphi_2(X) \not\subset (L^{(j)}_{\varphi_1(P)}\varphi_1(X))'$, and consider b_r with $r = s_j + 1$. From the maximality of s_j it follows that $b_r \neq 0$, as in the proof of Theorem 1. Therefore $\binom{m}{\mu_r} \not\equiv 0(p)$ and we are done since

$$\binom{m}{[\delta_j/q]} = \binom{m}{m-\mu_r} = \binom{m}{\mu_r}.$$

Conversely, if $\binom{m}{m-[\delta_j/q]} = \binom{m}{[\delta_j/q]} \not\equiv 0(p)$, then $m - [\delta_j/q]$ is an order of $\varphi_2(X)$ since m is. Therefore, by the definition of s_j, we must have $\mu_{s_j+1} = m - [\delta_j/q]$. Again $b_r \neq 0$ with $r = s_j + 1$ and this means that $L^{(s_j+1)}_{\varphi_2(P)}\varphi_2(X) \not\subset (L^{(j)}_{\varphi_1(P)}\varphi_1(X))'$, as was to be shown.

We now proceed to prove the claim. For $r = 0$ the claim follows from $m \not\equiv 0(p)$ and $b_0 \neq 0$. For $r = d+1$ it follows from $\binom{m}{m} = 1 \not\equiv 0(p)$ and $\sum_{i=0}^{n}(D^{(m)}y_i)^q \cdot x_i \neq 0$. So we can assume that $0 < \mu_r < m$. Applying the operator $D^{(\mu_r q)}$ to

$$\sum_{i=0}^{n} y_i^q \cdot D^{(q(m-\mu_r))} x_i = 0,$$

we get:

$$b_r + \sum_{k=1}^{\mu_r} \binom{m-\mu_r+k}{m-\mu_r} \sum_{i=0}^{n} (D^{(\mu_r-k)} y_i)^q \cdot D^{(q(m-\mu_r+k))} x_i = 0.$$

If $\mu_r - k$ is not an order of the morphism φ_2 then

$$\sum_{i=0}^{n} \left(D^{(\mu_r-k)} y_i\right)^q \cdot D^{(q(m-\mu_r+k))} x_i = 0,$$

since we can write the vector $\left(D^{(\mu_r-k)} y_0, \ldots, D^{(\mu_r-k)} y_n\right)$ as linear combinations of $\left(D^{(h)} y_0, \ldots, D^{(h)} y_n\right)$ with $h < \mu_r - k$. We conclude then that:

$$b_r + \sum_{\ell=0}^{r-1} \binom{m-\mu_\ell}{m-\mu_r} b_\ell = 0.$$

Assume, initially, that $b_r \neq 0$. From the equation above one gets that there exists $\ell_1 < r$ with $b_{\ell_1} \neq 0$ and $\binom{m-\mu_{\ell_1}}{m-\mu_r} \not\equiv 0$. If $\ell_1 > 0$, the same reasoning can be applied to b_{ℓ_1} yielding that there exists $\ell_2 < \ell_1 < r$ with $b_{\ell_2} \neq 0$ and $\binom{m-\mu_{\ell_2}}{m-\mu_{\ell_1}} \not\equiv 0$. Hence $\binom{m-\mu_{\ell_2}}{m-\mu_r} \not\equiv 0(p)$. After a finite number t of steps, one gets $\ell_t = 0$ and hence $\binom{m}{m-\mu_r} = \binom{m}{\mu_r} \not\equiv 0(p)$.

Conversely, we now show that $b_r = 0$ implies that $\forall \ell, 0 \leq \ell < r$ with $b_\ell \neq 0$, we have $\binom{m-\mu_\ell}{m-\mu_r} \equiv 0(p)$. In particular (for $\ell = 0$), we get $\binom{m}{\mu_r} \equiv 0(p)$. In fact, applying the operator $D^{(q)}$ to

$$0 = \sum_{i=0}^{n} \left(D^{(\mu_r-1)} y_i\right)^q \cdot D^{(q(m-\mu_r))} x_i,$$

we get:

$$-\mu_r b_r = (m-\mu_r+1) \sum_{i=0}^{n} \left(D^{(\mu_r-1)} y_i\right)^q \cdot D^{(q(m-\mu_r+1))} x_i.$$

Since $b_r = 0$, we have that the right hand side in the above equality is also equal to zero.

Applying the operator $D^{(2q)}$ to

$$0 = \sum_{i=0}^{n} \left(D^{(\mu_r-2)} y_i\right)^q \cdot D^{(q(m-\mu_r))} x_i,$$

we get:

$$-\binom{\mu_r}{2}b_r-(\mu_r-1)(m-\mu_r+1)\sum_{i=0}^{n}\left(D^{(\mu_r-1)}y_i\right)^q\cdot D^{(q(m-\mu_r+1))}x_i=$$
$$=\binom{m-\mu_r+2}{m-\mu_r}\sum_{i=0}^{n}\left(D^{(\mu_r-2)}y_i\right)^q\cdot D^{(q(m-\mu_r+2))}x_i.$$

Hence the right hand side in the above equality is also equal to zero. Continuing this process, we conclude that:

$$\binom{m-\mu_\ell}{m-\mu_r}b_\ell=0,\qquad \text{for } 0\le \ell<r.$$

This finishes the proof of Theorem 2. □

It is convenient to restate Theorems 1 and 2 for the case of an embedded curve, as follows:

Corollary 1. *Let $X\subseteq \mathbf{P}^n$ be an irreducible, non- degenerate, curve with orders $\varepsilon_0<\cdots<\varepsilon_n=mq$, where $m\not\equiv 0(p)$ and q is a power of p. Suppose that the dual curve X' is also non-degenerate and denote its orders by $\varepsilon'_0<\cdots<\varepsilon'_n$. Then, $m=\varepsilon'_{d+1}$ for some $d<n$ and defining s_j, $j=0,\ldots,n-1$, by $\varepsilon'_{s_j}<m-[\varepsilon_j/q]\le\varepsilon'_{s_{j+1}}$, we have $L^{(s_j)}_{\gamma(P)}X'\subseteq(L^{(j)}_PX)'$ for P generic, where $\gamma\colon X\to X'$ is the Gauss map. Moreover, if $\binom{m}{r}\not\equiv 0(p)$, then $qr=\varepsilon_j$ for some j, $\varepsilon'_{s_{j+1}}=m-r$ and $L^{(s_j+1)}_{\gamma(P)}X'\not\subseteq(L^{(j)}_PX)'$ for P generic.*

Proof. The first part is immediate from Theorem 1. For the second part recall that if $\binom{\varepsilon_n}{qr}\not\equiv 0(p)$ then qr is an order of X. As $\binom{\varepsilon_n}{qr}=\binom{mq}{rq}\equiv\binom{m}{r}(p)$, the result now follows from Theorem 2. □

The following two examples pertain to the above corollary. The first shows that the corollary's hypotheses can hold even in a very pathological situation and the second shows that they may not always hold.

Example 1. Let $1<q_1<q_2<q_3$ be powers of the characteristic p. Let $X\subseteq\mathbf{P}^4$ be given by

$$\left(1:t:t^{q_1}:t^{q_2}:t^{q_2+1}+t^{q_1+q_3}\right).$$

Then X has orders $0,1,q_1,q_2,q_2+1$, the dual curve X' is non-degenerate with orders $0,1,q_2,q_2+1,q_3$, but X'' is degenerate and, in particular, $X\neq X''$. The integers s_j, $j = 0,1,2,3$, are given by: $s_0 = 2$, $s_1 = s_2 = 1$ and $s_3 = 0$. The inclusions relating osculating spaces (given in Corollary 1) are the best possible since, in this case, we have that $\binom{m}{\varepsilon'_j} \not\equiv 0(p)$, for $0 \leq j \leq d+1 = 3$.

Example 2. It is not always the case that $\binom{m}{\varepsilon'_j} \not\equiv 0(p)$, for $0 \leq j \leq d+1$. Take $p = 3$ and $X \subseteq \mathbf{P}^7$ given by

$$\left(1{:}\, t{:}\, t^3{:}\, t^4{:}\, t^6{:}\, t^9{:}\, t^{10} + t^{13}{:}\, t^{12} + t^{13} + t^{15}\right).$$

Then X has orders 0,1,3,4,6,9,10,12, so $m = 4$. It can be checked that X' is non-degenerate and has 2 as an order. Still, $\binom{4}{2} \equiv 0(3)$.

The following corollary shows, in particular, the biduality of curves in characteristic zero (see Remark 2 below).

Corollary 2. *Notation and hypotheses as in Corollary 1. If for some t, $\varepsilon_{n-i} = (m-i)q$ and $\binom{m}{i} \not\equiv 0(p)$ hold for $i < t$, then $\varepsilon'_i = i$ and $L^{(i)}_{\gamma(P)}X' = \left(L_P^{(n-1-i)}X\right)'$ hold for $i < t$. If, further, $\binom{m}{t} \not\equiv 0(p)$ but $\varepsilon_{n-t} \neq (m-t)q$, then $\varepsilon'_t = t$ but $L^{(t)}_{\gamma(P)}X' \neq \left(L_P^{(n-1-t)}X\right)'$.*

Proof. Let $\varepsilon'_{d+1} = m$ as in Corollary 1. Since $\binom{\varepsilon'_{d+1}}{i} \not\equiv 0(p)$ for $i < t$, it follows that $\varepsilon'_i = i$ for $i < t$. From Corollary 1, $s_{n-i} = i-1$ for $i < t$. As $\dim \left(L_P^{(n-1-i)}X\right)' = i = \dim L^{(i)}_{\gamma(P)}X'$, the equality of the spaces follows.

If $\binom{m}{t} \not\equiv 0(p)$ then, as before, $\varepsilon'_t = t$ and $(m-t)q$ is an order of X. Since $\varepsilon_{n-t} \neq (m-t)q$, we must have $(m-t)q = \varepsilon_r$ for some $r < n-t$. Hence $s_r = t-1$ and the result follows again from Corollary 1. □

Remark 2. If $\varepsilon_i = i$, $i = 0,1,\dots,n$, and $p = 0$ or $p > n$, then the hypotheses of Corollary 2 are satisfied with $q = 1$ and $t = n+1$. It follows that X' is classical and that $L^{(i)}_{\gamma(P)}X' = \left(L_P^{(n-1-i)}X\right)'$, for $i = 0,1,\dots,n$. In particular, for $i = n-1$, $L^{(n-1)}_{\gamma(P)}X' = P'$; so $X'' = X$ and the map $\gamma' \circ \gamma{:}\, X \to X$ is the identity, where $\gamma'{:}\, X' \to X''$ is the Gauss map of X'.

For a matrix (A_{ij}), $0 \leq i \leq n-1$ and $0 \leq j \leq n$, we denote by $(A_{ij})_{j \neq n-\ell}$ the square matrix obtained by deleting the $(n-\ell)$- th column of the matrix (A_{ij}).

It is useful to have a relation among the orders of a curve and those of its dual. The following is a result on this line.

Proposition 1. *Let $X \subseteq \mathbf{P}^n$ be an irreducible, non-degenerate curve with orders $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_n$. Suppose that the dual curve X' is also non-degenerate with orders $\varepsilon'_0 < \varepsilon'_1 < \cdots < \varepsilon'_n$. If $\varepsilon_n \not\equiv 0(p)$, then $\varepsilon'_\ell \geq \varepsilon_n - \varepsilon_{n-\ell}$, for $\ell = 0, 1, \ldots, n$. Moreover, for any given ℓ, $0 \leq \ell \leq n$, we have:*

(i) *If $\varepsilon'_\ell = \varepsilon_n - \varepsilon_{n-\ell}$, then* $\det\left(\binom{\varepsilon_j}{\varepsilon_i}\right)_{j \neq n-\ell} \not\equiv 0\ (p)$

(ii) *If* $\det\left(\binom{\varepsilon_j}{\varepsilon_i}\right)_{j \neq n-\ell} \not\equiv 0\ (p)$, *then $\varepsilon_n - \varepsilon_{n-\ell}$ is an order of X'.*

Proof. Let t be an uniformizing parameter at an ordinary point on X. Choose coordinates x_j in $\mathbf{P}^n$ such that $x_j = (t^{\varepsilon_j} +$ higher terms $)$ on the curve X. The dual curve is then traced by $(\alpha_0 : \cdots : \alpha_n)$, where α_ℓ is given by

$$\alpha_\ell = \det\left(D_t^{(\varepsilon_i)} x_j\right)_{j \neq n-\ell} = \det\left(\begin{pmatrix} \varepsilon_j \\ \varepsilon_i \end{pmatrix}\right)_{j \neq n-\ell} \cdot t^{\varepsilon_n - \varepsilon_{n-\ell}} + \text{ higher terms.}$$

The proposition now follows. □

Remark 3. Suppose that $\varepsilon'_n = \varepsilon_n \not\equiv 0(p)$. Suppose moreover that $\binom{\varepsilon_n}{\varepsilon_\ell} \not\equiv 0(p)$, for all $\ell = 0, 1, \ldots, n$. Then we have that $\binom{\varepsilon'_n}{\varepsilon'_\ell} = \binom{\varepsilon'_n}{\varepsilon_n - \varepsilon_\ell} \not\equiv 0(p)$, $\forall \ell$. It now follows that $\varepsilon'_\ell = \varepsilon_\ell = \varepsilon_n - \varepsilon_{n-\ell}$, $\forall \ell = 0, 1, \ldots, n$. From Proposition 1, we conclude that $\det\left(\binom{\varepsilon_j}{\varepsilon_i}\right)_{j \neq n-\ell} \not\equiv 0(p)$, $\forall \ell$. In general, for a fixed $\ell \in \{0, 1, \ldots, n\}$, one can check that $\det\left(\binom{\varepsilon_j}{\varepsilon_i}\right)_{j \neq n-\ell}$ is a multiple of $\binom{\varepsilon_n}{\varepsilon_{n-\ell}}$, and hence $\det\left(\binom{\varepsilon_j}{\varepsilon_i}\right)_{j \neq n-\ell} \not\equiv 0(p)$ implies $\binom{\varepsilon_n}{\varepsilon_{n-\ell}} \not\equiv 0(p)$.

Remark 4. If $\varepsilon_i = i$ $(i = 0, 1, \ldots, n)$ and $p < n$, then Proposition 1 gives that

$$\varepsilon'_\ell = \ell, \quad \text{for all } \ell \text{ if and only if } \quad \det\left(\begin{pmatrix} j \\ i \end{pmatrix}\right)_{j \neq n-\ell} \not\equiv 0(p), \text{ for all } \ell.$$

Direct computations show that the determinants above are equal to the binomial coefficients $\binom{n}{\ell}$. Hence the dual curve X' is also classical if and only if the integer n has the following p-adic expansion:

$$n = n_0 p^r + (p-1)p^{r-1} + (p-1)p^{r-2} + \cdots + (p-1), \quad 1 \leq n_0 < p.$$

This provides a way of constructing non-classical curves by dualizing classical ones in small characteristics.

We end up this section characterizing the non-degenerate curves with degenerate duals. Recall that a curve is said to be *strange* if all tangents to smooth points are concurrent. Note that a strange curve has a degenerate dual.

Proposition 2. *Let $X \subseteq \mathbf{P}^n$ be an irreducible, non-degenerate and non-strange curve. Let $\varepsilon_0 < \cdots < \varepsilon_{n-1} < \varepsilon_n$ be the order-sequence of X. The dual of X is degenerate if and only if the curve X lies on a cone over a curve $C \subseteq \mathbf{P}^{n-1}$ with orders $\delta_0 < \cdots < \delta_{n-1}$ satisfying $\delta_{n-1} > \varepsilon_{n-1}$.*

Proof. First notice that any projection of X from a point P onto a curve $C \subseteq \mathbf{P}^{n-1}$ is separable since, otherwise, all tangents to X would pass through P and the curve X would be strange. Hence, a separating variable on any such a curve C is also a separating variable on X. Also, any such a curve $C \subseteq \mathbf{P}^{n-1}$ is non-degenerate since $X \subseteq \mathbf{P}^n$ is.

Consider the projection $C \subseteq \mathbf{P}^{n-1}$ of X from the point $(0:0:\cdots:0:1)$ and its order sequence $\delta_0 < \delta_1 < \cdots < \delta_{n-1}$. The proposition follows from the equivalence of the following conditions:

(i) All the osculating hyperplanes to X pass through the point $(0:0:\cdots:0:1)$.

(ii) $\det\left(D^{(\varepsilon_i)}x_j\right)_{0\leq i,j\leq n-1} = 0.$

(iii) $\delta_{n-1} > \varepsilon_{n-1}$.

The equivalence of (i) and (ii) follows from the equations for the osculating hyperplanes at general points of X (see [17] Corollary 1.3).

The equivalence of (ii) and (iii) follows from the definition of order-sequence as the minimal sequence in the lexicographic order making certain wronskian determinants not identically zero (see [17] page 5) and from the trivial fact that if vectors in K^{n+1} are linearly dependent over K, then the vectors in K^n obtained by deleting the last coordinates are also linearly dependent over K. □

This result has been recently used by Homma to prove that, in $\mathbf{P}^3$, the dual of X is degenerate if and only if the tangent developable of X is strange.

4. Biduality for Projective Curves

In this section we study the bidual of a curve and, in particular, conditions for it to coincide with the curve. The basic result is:

Theorem 3. *Let $X \subseteq \mathbf{P}^n$ be an irreducible non-degenerate curve with order-sequence $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_n$. Suppose that the dual curve X' is also non-degenerate and denote its order-sequence by $\varepsilon'_0 < \cdots < \varepsilon'_n$. For a morphism $F: X \to X$ satisfying $deg_i F = q deg_i \gamma$, where $q \geq 1$ and $\gamma: X \to X'$ is the Gauss map, the following conditions are equivalent:*

1) *$X = X''$ and $F = \gamma' \circ \gamma$, where $\gamma': X' \to X''$ is the Gauss map.*

2) *There exists a natural number m such that $\varepsilon'_n = mq$ and such that $L^{(d)}_{F(P)} X \subseteq L^{(n-1)}_P X$ for P generic, where the integer d is defined by $\varepsilon_d < m \leq \varepsilon_{d+1}$.*

Furthermore, in this case, $\varepsilon_{d+1} = m \not\equiv 0(p)$, $L^{(d+1)}_{F(P)} X \not\subset L^{(n-1)}_P X$ and, defining s_j ($j = 0, \ldots, n-1$) by $\varepsilon_{s_j} < m - [\varepsilon'_j / q] \leq \varepsilon_{s_{j+1}}$, we have $L^{(j)}_{\gamma(P)} X' \subseteq \left(L^{(s_j)}_{F(P)} X \right)'$, for P generic.

Proof. Assume that 1) holds. Then we have that $\deg_i \gamma' = q$ and hence that $\varepsilon'_n = mq$, for some integer m. Apply now Theorem 1 with $\varphi_1 = \gamma$ and $\varphi_2 = F$. The converse also follows directly from Theorem 1. □

The following three examples pertain to the theorem above. In the first two examples the hypotheses are satisfied and the last one shows their necessity.

Example 3. This is an example of a curve $X \subseteq \mathbf{P}^3$ such that, generically, the tangent line at the image under a Frobenius morphism of a point is contained in the osculating plane at that point. For that, consider the curve $Y \subseteq \mathbf{P}^3$ given by

$$x^{q+1} + y^{q+1} = 1 \quad \text{and} \quad x^{q+1} + z^{q+1} = 2,$$

in affine coordinates x, y and z. It is easy to check that Y has orders $0, 1, q, 2q$ and that $F(P) \in T_P Y$ for P generic, where F is the Frobenius morphism of degree q^2 and $T_P Y$ denotes the tangent line to Y at P. From Corollary 2, $\varepsilon'_i = i$ for $i \leq 2$. We also have that $\varepsilon'_3 = q$ and hence, applying Theorem 3, that $Y = Y''$. Let now $X = Y'$. So $X = X''$ and, applying the other implication in Theorem 3, we see that X has the desired property.

Example 4. Let C be a non-classical plane curve with orders 0, 1, q'. For an integer $m < q'$, let Φ_m be the m-uple embedding $\Phi_m: C \to \mathbf{P}^N$, $N = \binom{m+2}{2} - 1$, and let $X = \Phi_m(C)$. The set of orders of the curve X is $\{r + sq' | r + s \leq m\}$. In particular, $\varepsilon_N = mq'$ and the equation of the osculating hyperplane of X at the point $\Phi_m(Q)$ is the m-th power of the equation of T_QC, where T_QC denotes the tangent line to C at the point Q. If the plane curve C is such that $F(Q) \in T_QC$ for Q generic (see [3], [4] and [9]), where F is the Frobenius map of degree qq' with $q > 1$, then

$$L_{F(P)}^{(m-1)}X \subseteq L_P^{(N-1)}X \quad \text{and} \quad L_{F(P)}^{(m)}X \not\subseteq L_P^{(N-1)}X, \quad \text{for} \quad P \quad \text{generic} .$$

It is easy to check that X' is degenerate if and only if $m \geq p$. If $m < p$, a direct computation shows that X' is equal to $\Phi_m(C')$ up to an automorphism of $\mathbf{P}^N$ induced by a diagonal matrix, and hence the orders of X' are given by $\{r + sq | r + s \leq m\}$. This provides lots of examples for Theorem 3.

Example 5. Let $q > 1$ and $q' > 1$ be powers of the characteristic of the ground field. Let C be an algebraic curve and x, u, v, w be separating variables on C. Consider the map $\Phi: C \to \mathbf{P}^3$ given by

$$(1: x: x^2: u^{q'} + v^{q'}x + w^{q'}x^2)$$

and let $X = \Phi(C)$. The order-sequence of $X \subseteq \mathbf{P}^3$ is 0, 1, 2, q' and the dual curve X' is given by $(u^{q'}: v^{q'}: w^{q'}: -1)$. So the dual X' can be anything and we cannot control its order-sequence in general.

To specialize this example further, let $g(x, y)$ be an irreducible factor of the polynomial

$$(y^{qq'} - y)x^{2q} + x^q\left((x^{qq'} - x) + (x^{qq'} - x)^q\right) + (x^{qq'} - x)$$

which is not a linear combination of $1, x, y$ and $x + x^q$ (it is not hard to see that such a $g(x, y)$ exists). Taking C as the plane curve given by $g(x, y) = 0$ and considering the embedding Φ of C in $\mathbf{P}^3$ above, with $u = x$, $v = x + x^q$ and $w = y$, we see that all hypotheses in Theorem 3 (with $m = 1$ and F being the Frobenius map of degree qq') are satisfied, because of the way $g(x, y)$ was chosen, except that $\varepsilon_2' = q$ holds instead of $\varepsilon_3' = q$; a direct calculation shows that $X \neq X''$.

In Remark 2 it was observed that, in characteristic zero, we always have that $X'' = X$ and also that $\gamma' \circ \gamma$ is the identity. The following corollary characterizes those curves in positive characteristic for which this also happens.

Corollary 3. *Let $X \subseteq \mathbf{P}^n$ be an irreducible, non- degenerate curve with orders $\varepsilon_0 < \cdots < \varepsilon_n$. Suppose that X' is also non-degenerate and denote* *by* $\varepsilon'_0 < \cdots < \varepsilon'_n$ *its orders. If* $\varepsilon_n \cdot \varepsilon'_n \not\equiv 0(p)$, *then* $X'' = X$ *if and only if* $\varepsilon'_n \leq \varepsilon_n$. *If this is the case, then* $\gamma' \circ \gamma : X \to X''$ *is the identity map and* $\varepsilon'_n = \varepsilon_n$.

Proof. If $\varepsilon'_n \leq \varepsilon_n$, then $m = \varepsilon'_n \leq \varepsilon_n$. We can thus apply Theorem 3 with F being the identity map (and $d = n-1$). Reciprocally, if $X'' = X$, then $\gamma' \circ \gamma$ is separable since $\varepsilon_n \cdot \varepsilon'_n \not\equiv 0(p)$. So $q = 1$ and $\varepsilon'_n = m$ and, by Theorem 3 with $F = \gamma' \circ \gamma$, we get that $L^{(d)}_{F(P)}X \subseteq L^{(n-1)}_P X$. In particular, $d \leq n-1$, that is $\varepsilon'_n \leq \varepsilon_{d+1} \leq \varepsilon_n$. This proves the first part of the corollary.

As for the second part, $\varepsilon_n = \varepsilon'_n$ since applying the first part with the roles of X and X' reversed we get $\varepsilon'_n \geq \varepsilon_n$. Also, Theorem 3 can be applied with F being the identity map and so, necessarily, $\gamma' \circ \gamma$ is the identity.□

The separability of $\gamma' \circ \gamma$ is not enough to guarantee that $X'' = X$, as shown by the following.

Example 6. The curve $X \subseteq \mathbf{P}^5$ given by

$$(1 : t : t^2 : t^3 : t^q + t^{2q+3} : t^{q+1} + t^{2q+2})$$

is non-degenerate and has orders 0, 1, 2, 3, $q, q+1$. The dual curve X' is also non-degenerate and has orders 0, 1, $q, q+1, 2q, 2q+1$, as one can easily check. This gives an example of a curve with $\varepsilon_n \cdot \varepsilon'_n \not\equiv 0(p)$ but with $X \neq X''$.

Corollary 4. *Suppose that* $X = X''$ *and that* $\gamma' \circ \gamma$ *is the identity map. Suppose moreover that*

$$\binom{\varepsilon_n}{\varepsilon_\ell} \not\equiv 0(p), \ \forall \ell = 0, 1, \ldots, n-1.$$

Then, $\varepsilon_\ell = \varepsilon'_\ell = \varepsilon_n - \varepsilon_{n-\ell}$, $\quad \forall \ell = 0, 1, \ldots, n$, *and*

$$L^{(n-1-j)}_{\gamma(P)} X' = \left(L^{(j)}_P X\right)'$$

for P *generic and* $\forall j = 0, 1, \ldots, n-1$.

Proof. Since $\gamma' \circ \gamma$ is separable, we have $\varepsilon_n \varepsilon'_n \not\equiv 0(p)$. From Corollary 3, we get that $\varepsilon_n = \varepsilon'_n$. The equality $\varepsilon_\ell = \varepsilon'_\ell = \varepsilon_n - \varepsilon_{n-\ell}$ now follows from Remark 3. The integer s_j defined by

$$\varepsilon'_{s_j} < \varepsilon_n - \varepsilon_j \leq \varepsilon'_{s_j+1} \quad \text{is such that} \quad \varepsilon'_{s_j+1} = \varepsilon_n - \varepsilon_j,$$

since $\varepsilon_n - \varepsilon_j = \varepsilon_{n-j} = \varepsilon'_{n-j}$, and hence $s_j = n - 1 - j$. The equality relating osculating spaces follows from Corollary 1, noticing that $\dim \left(L_P^{(j)} X\right)' = n - 1 - j$.□

5. Biduality for Plane Curves

A plane curve is said to be *reflexive* if its Gauss map is separable. It is well-known that, for reflexive curves X, we have $X'' = X$ and $\gamma' \circ \gamma$ is the identity. This also follows from the results of §4 since reflexive curves do not exist in characteristic two and they always have $\varepsilon_2 = 2$ in the other characteristics.

Suppose that X is a non-reflexive plane curve but that $X'' = X$. By Theorem 3, there exists an inseparable map $F: X \to X$ satisfying $F(P) \in T_P X$ ($T_P X$ is the tangent line to X at P) for all smooth points $P \in X$. If, moreover, F is purely inseparable (for example if X has genus at least two), then F is abstractly a Frobenius map (see [5] IV prop. 2.5). In general circumstances one can guarantee that F is induced by the Frobenius map in $\mathbf{P}^2$, as shows the following proposition.

Proposition 3. *Let X be a smooth plane curve of degree at least four and $F: X \to X$ an inseparable morphism. Then, for some system of coordinates in the plane, the morphism F is induced by a Frobenius map in $\mathbf{P}^2$.*

Proof. Since X has genus at least two, F is purely inseparable. Let q be the degree of F. Let also $(x_0: x_1: x_2)$ be coordinates for $X \subseteq \mathbf{P}^2$, where x_i, $i = 0, 1, 2$, are functions on X. Write $F(x_i) = y_i^q$, $i = 0, 1, 2$, for some functions y_i on X. The morphism $X \to \mathbf{P}^2$ given by $(y_0: y_1: y_2)$ is clearly of the same degree as $(x_0: x_1: x_2)$ and so (by [1] app A, Ex 18, p. 56) it differs from $(x_0: x_1: x_2)$ by a linear change of coordinates, that is,

$$y_i = \sum_{j=0}^{2} a_{ij} x_j, \quad i = 0, 1, 2,$$

and the matrix (a_{ij}) is invertible. If we change coordinates in $\mathbf{P}^2$ by some invertible matrix (c_{ij}), then the matrix (a_{ij}) changes to

$$(c_{ij}^{1/q})(a_{ij})(c_{ij})^{-1},$$

which can be made equal to the identity by a suitable choice of the matrix (c_{ij}). In this case F is just the $\mathbf{F}_q$-Frobenius map, and this proves the proposition. □

Remark 5. An analogue of Proposition 3 also holds if, instead of a smooth plane curve, one has a *canonical* curve, that is, a curve of degree $2g-2$ and of genus $g \geq 3$ in $\mathbf{P}^{g-1}$.

A plane curve defined over the finite field $\mathbf{F}_q$ is called *Frobenius non-classical* if $F(P) \in T_P X$, for all smooth points P on X , where F is the $\mathbf{F}_q$-Frobenius map. For a Frobenius non-classical curve, it is shown in [9] that the Gauss map $\gamma: X \to X'$ is purely inseparable of degree $q' \leq q$ and, moreover, $q' < q$ if X is smooth.

The dual of a plane curve X is degenerate if and only if the tangents to smooth points of X are concurrent, that is, X is strange. Recall that there are no smooth strange curves of degree at least three.

Putting all these results together with Theorem 3, we get:

Theorem 4. *Let X be a smooth non-reflexive plane curve of degree at least four. Then $X'' = X$ if and only if the curve X is defined over a finite field and it is Frobenius non-classical.*

Remark 6. A large class of examples of smooth plane curves X with $X'' = X$, but with inseparable Gauss maps, is thus given by the Fermat curves $x^d + y^d = 1$, where $d = (q-1)/(q'-1)$ and q is a power of q' (see [4], Theorem 2). See also [3] for other examples and [9] for an extensive study of Frobenius non-classical plane curves.

Remark 7. It is an elementary exercise to show that Theorem 4 is still valid for smooth conics and cubics.

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